Journal of Applied Mathematics and Mechanics 73 (2009) 204–208

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

The use of near-front ray expansions in deformation dynamics☆

Ye.A. Gerasimenko, A.V. Zavertan, V.Ye. Ragozina

Vladivostock, Russia

ARTICLE INFO

Article history:
Received 11 March 2008

ABSTRACT

A method is proposed for the constructing approximate solutions of problems in impact deformation dynamics in the form of a ray expansion of the solution at the strain discontinuity surface. The fundamental difference in the proposed approximate method is the fact that, when constructing the ray expansions of the solutions, account is taken of the regularities in the change in the curvatures of the surfaces of strain discontinuities and the divergence of the rays. The main qualitative features of the method are illustrated using the example of a one-dimensional cylindrical shock wave.

Approximate solutions of unsteady problems in the theory of elasticity,[2,3] viscoelasticity,[4] and ideal plasticity when the system of equations of the theory in Ref. 5 is hyperbolic are constructed using next-front ray expansions in deformation dynamics (see the review in Ref. 1). A ray expansion is a Taylor-type series, the argument of which is the distance along the ray from a moving discontinuity surface and the coefficients are the discontinuities in the strains and their derivatives on this surface. These coefficients are related to one another by the compatibility conditions for the discontinuities. When it is assumed that the deformations are small or that the deformations are continuous on the discontinuity surface (weak waves), ordinary differential equations for the discontinuities are found as a result of the compatibility conditions, that is, for the coefficients of the ray series. The construction of the approximate solutions is concluded by solving the corresponding Cauchy problem. This procedure is then found to be recurrent. Another situation arises when the shock wave, in the case of an essentially non-linear form of the deformation process, serves as the discontinuity surface. In this case, the damping equations do not follow from the conditions for the compatibility of the discontinuity. A proposal was introduced in Refs 6 and 7 for overcoming this fundamental difficulty by expanding the intensity of a discontinuity in a power series in time. The approximate solutions of the impact deformation boundary-value problems constructed by this method will be closer to the exact solutions the smaller the post-impact times considered.

Another version of the modification of the ray method is proposed below, in which additional information in included concerning the magnitude of the higher-order discontinuities in the recurrent system of damping equations. The source of this information is the linearized solution of the problem, which enables one to represent the dynamics of the change in the parameters of the initial action, as well as the change in the geometry of the leading edge of the wave, more precisely. This proposal is demonstrated for the simplest example of a one-dimensional cylindrical shock wave.

1. Model relations of a non-linearly elastic isotropic medium. Boundary conditions on shock waves

The behaviour of a non-linearly elastic material in an Eulerian curvilinear system of coordinates x^i ($i = 1, 2, 3$) is determined by the system of equations

$$\rho = \rho_0 \det(\delta^i_j - u^i_{,j}), \quad \upsilon^i = \dot{u}^i + u^i_{,j}\upsilon^j, \quad \sigma^{ij}_{,j} = \rho(\dot{\upsilon}^i + \upsilon^i_{,j}\upsilon^j)$$

$$2\alpha_{ij} = u_{i,j} + u_{j,i} - u_{k,i}u^k_{,j}, \quad \sigma^i_j = \frac{\rho}{\rho_0}\frac{\partial W}{\partial \alpha^j_k}(\delta^i_k - 2\alpha^i_k) \tag{1.1}$$

☆ *Prikl. Mat. Mekh*. Vol. 73, No. 2, pp. 282–288, 2009.

E-mail address: ekaterina_gerasi@mail.ru (Ye.A. Gerasimenko).

0021-8928/$ – see front matter
doi:10.1016/j.jappmathmech.2009.04.010

where ρ and ρ_0 are the density of the medium in the current and free state, u^i and u_i are the contravariant and covariant components of the displacement vector, υ^i are the contravariant components of the velocity of the points of the medium, α_{ij} and α^i_k are the covariant and mixed components of the Almansi strain tensor, σ^{ij} and σ^i_j are the contravariant and mixed components of the Euler–Cauchy stress tensor, W is a function of the elastic potential of the medium, δ^i_j is the Kronecker delta, partial differentiation with respect to time is denoted by a dot above a symbol and an index after a comma denotes the operation of covariant differentiation.

Assuming that the material is isotropic and neglecting thermal effects, the elastic potential function can be represented in the form of a Taylor series in the neighbourhood of the free state

$$W = \frac{\lambda}{2}I_1^2 + \mu I_2 + lI_1I_2 + mI_1^3 + nI_3 + \xi I_2^2 + \eta I_1^2 I_2 + \kappa I_1 I_3 + \chi I_1^4 + \ldots$$
$$I_1 = \alpha^i_i, \quad I_2 = \alpha^i_j\alpha^j_i, \quad I_3 = \alpha^i_j\alpha^j_k\alpha^k_i \tag{1.2}$$

Here λ and μ are Lamé parameters, and l, m, n, ξ, η, κ, λ are higher order elastic moduli.

The system of equations (1.1) holds everywhere in the case of impact deformation problems with the exception of the discontinuity surfaces where the large changes in the gradient of the displacements are replaced by its discontinuous representation. It is well known[8] that, in shock waves, the required fields satisfy dynamic compatibility conditions, following from the integral laws for the conservation of mass, momentum and energy:

$$[\rho(\upsilon^i\nu_i - G)] = 0, \quad [\sigma^{ij}]\nu_j = \rho^+(\upsilon^{j+}\nu_j - G)[\upsilon^i]$$
$$\sigma^{ij+}[\upsilon_i]\nu_j = \rho^+(\upsilon^{j+}\nu_j - G)\left(\frac{[\upsilon_i][\upsilon^i]}{2} - [e]\right) - [q^j]\nu_j \tag{1.3}$$

A discontinuity of a quantity in them is denoted by square brackets so that $[f] = f^+ - f^-$ where f^+ and f^- are the limit values of f as the discontinuity surface Σ is approached from the two different sides, ν_i are the components of the unit outward normal vector, directed in the sense of the motion of Σ (in the domain V^+, G is the velocity of the surface Σ in the direction of the normal), q_i are the components of the heat flux vector and e is the internal energy density.

We further assume that the discontinuities in the surface Σ are regular in the sense that they satisfy the geometric and kinematic compatibility conditions

$$[f_{,i}] = \left[\frac{\partial f}{\partial \nu}\right]\nu_i + g^{\alpha\beta}[f]_{,\beta}g_{ij}x^j_{,\alpha}, \quad [\dot{f}] = -G\left[\frac{\partial f}{\partial \nu}\right] + \frac{\delta[f]}{\delta t} \tag{1.4}$$

The components of a tensor field, defined in the whole of the volume can be taken as f, $[f]_{,\beta}$ is the tensor derivative of this field in the surface Σ in the sense of the known definition,[9] g_{ij} are the covariant components of the spatial metric tensor and $g^{\alpha\beta}$ are the contravariant components of the surface metric tensor.

The possible velocities of the shock waves can be obtained on the basis of relations (1.3) and (1.4) and the nature of the change in the deformation fields in them can be indicated.[10] Note that, for non-linear processes in the general case, action on a medium is transmitted by three waves: a quasilongitudinal wave and two quasitransverse waves [10]. The velocities of these waves are found to be functions of the vector for the intensity of the discontinuity and prior deformations. In the general case, the solution of the boundary value problems therefore not only involves the definition of the strain fields, deformations and stresses behind the wave fronts but, also, the definition of the geometric characteristics of these fronts.

2. Ray solution of the one-dimensional problem of a diverging cylindrical shock wave

We will consider the one-dimensional problem of a longitudinal cylindrical shock wave as a model problem which allows the main features of the technique proposed later to be shown. As a result of a normal action on a surface L_0 (it corresponds to the boundary of a cylindrical sheet in an unbounded space or it is the boundary of a cylinder of initial radius r_0), a diverging or converging longitudinal shock wave arises in the medium from the initial instant. The displacement field is such that $u_r = u_r(r; t)$, $u_\varphi = u_z = 0$, where r, ν and z are cylindrical coordinates and u_r, u_φ, u_z are the physical components of the displacement vector. In this case,

$$u_{r,rr}\left(1 + \alpha_1 u_{r,r} + \alpha_2\frac{u_r}{r}\right) + \frac{u_{r,r}}{r} - \frac{u_r}{r^2} + \alpha_3\frac{u_r^2}{r^3} + \alpha_4\frac{u_{r,r}^2}{r} + \alpha_5\frac{u_r u_{r,r}}{r^2} + \ldots =$$
$$= C^{-2}\left\{\ddot{u}_r\left(1 - 2u_{r,r} - \frac{u_r}{r}\right) + 2\dot{u}_r\dot{u}_{r,r}\right\} + \ldots$$
$$\alpha_1 = -9 + 6\frac{l+m+n}{\lambda+2\mu}, \quad \alpha_2 = -\frac{4\lambda+2\mu}{\lambda+2\mu} + \frac{2l+6m}{\lambda+2\mu}$$
$$\alpha_3 = \frac{5\lambda+7\mu}{\lambda+2\mu} - \frac{4l+6m+3n}{\lambda+2\mu}, \quad \alpha_4 = -3-\alpha_3, \quad \alpha_5 = 3, \quad C^2 = \frac{\lambda+2\mu}{\rho_0} \tag{2.1}$$

will be a consequence of system (1.1). The displacement field in the boundary $L(t)$ is known and can be specified by a Taylor series

$$u_r|_{r_L} = g(t), \quad r_L = r_0 + g(t)$$

$$g(t) = \sum_{k=1}^{\infty} \frac{1}{k!}\frac{d^k g(0)}{dt^k} t^k = \upsilon_0 t + \frac{at^2}{2} + \ldots, \quad \upsilon_0 \neq 0, \quad t \geq 0 \tag{2.2}$$

It is subsequently possible to restrict this to a quadratic representation which is only done to shorten the calculations presented here to some extent and is not fundamental. When there has been no earlier deformation of the medium, a velocity of the discontinuity surface $\Sigma(t)$, that is, of the leading edge of the longitudinal shock wave, given by the formula

$$G = C\sum_{k=0}^{\infty} \beta_k \tau^k \approx C(1 + \beta_1\tau + \beta_2\tau^2), \quad \tau = [u_{r,r}] = u^+_{r,r} - u^-_{r,r}$$

$$\beta_0 = 1, \quad \beta_1 = \frac{9}{4} - \frac{3(l+m+n)}{2(\lambda+2\mu)} = -\frac{\alpha_1}{4}$$

$$\beta_2 = \frac{9}{4} - 6\frac{l+m+n}{\lambda+2\mu} + 2\frac{\chi+\xi+\eta+\kappa}{\lambda+2\mu} - \frac{(1-\beta_1)^2}{2} \tag{2.3}$$

will be a consequence of relations (1.3) and (1.4) in the case being considered.

The boundary conditions on the surface $\Sigma(t)$ have the form

$$u_r|_{r_\Sigma} = 0, \quad \tau|_{r_\Sigma} = -u^-_{r,r}; \quad r_\Sigma = r_0 \pm \int_0^t G(\xi)d\xi \tag{2.4}$$

The plus sign corresponds to a diverging shock wave and the minus sign to a converging shock wave.

The boundary value problem (2.1)–(2.4) cannot be solved exactly in view of the non-linearity of the equation of motion and the boundary conditions. Attention is also drawn to the fact that conditions (2.4) are specified on the moving surface $r_\Sigma(t)$, the position of which is unknown in advance and is one of the quantities which are determined in the course of the solution. Note that, in the general case, the geometry of the wave surface also turns out to be unknown but, in the simple case considered here, only the position of the cylindrical discontinuity surface turns out to be unknown. The method of ray expansions is then used to determine the solution.

We assume that, in the zone in front of and behind the wave adjoining the surface $\Sigma(t)$, the displacement field is fairly smooth and permits up to an arbitrary k-th order partial differentiation with respect to time. We represent the required solution for $u_r(r;\, t)$ by the ray series

$$u_r(r;\, t) = \begin{cases} u_r^0(r;\, t) - \displaystyle\sum_{k=1}^{\infty} \frac{1}{k!}\left[\frac{\partial^k u_r}{\partial t^k}\right]\Bigg|_{t_\Sigma(r)} (t - t_\Sigma(r))^k, & t \geq t_\Sigma \\ u_r^0(r;\, t), \quad t \leq t_\Sigma & \end{cases}$$

$$t_\Sigma = \pm\int_{r_0}^{r} \frac{d\xi}{G(\xi)}, \quad \eta_i = \left[\frac{\partial^i u_r}{\partial t^i}\right] \tag{2.5}$$

where $u_r^0(r;\, t)$ is the given known displacement field in front of the wave, $u_r^0(r;\, t) = 0$ in the case considered and $t_\Sigma(r)$ is the eikonal equation. The series (2.5) is similar in type to a Taylor series but its coefficients η_i are calculated on the moving surface $\Sigma(t)$. If it is assumed that the equations of motion not only hold in a small neighbourhood of the surface $\Sigma(t)$ and the consequences of their differentiation up to k-times also hold, then, by writing the resulting equations in discontinuities on the surface $\Sigma(t)$, we obtain the recurrence relations

$$\frac{\delta\eta_{i+1}}{\delta t} = f_{i+1}(\eta_1, \eta_2, \ldots, \eta_{i+2}, r_\Sigma(t)), \quad i = 0, 1, \ldots, k \tag{2.6}$$

where $i = 0$ corresponds to the initial equations of motion in the discontinuities. If acceleration waves were considered, then the quantity η_{i+2} would not appear among the arguments in relations (2.6). In this case, equalities (2.6) can be considered as a system of ordinary differential equations, and the values of η_i would be determined by its successive integration. The existence of the quantity η_{i+2} as an argument in system (2.6) is the distinctive feature of the shock wave. It is regarded as a limitation on the applicability of the ray method.

It has already been mentioned that a version of the ray method has been proposed [6,7] in which additional expansions in δ-derivatives[11,12] for small post-impact times are constructed for the quantities η_i

$$\eta_i \approx \eta_{i0} + \frac{\delta\eta_{i0}}{\delta t}t + \frac{1}{2}\frac{\delta^2\eta_{i0}}{\delta t^2}t^2 + \ldots, \quad \frac{\delta^k\eta_i}{\delta t^k}\bigg|_{t=0} = \frac{\delta^k\eta_{i0}}{\delta t^k} \tag{2.7}$$

This enables us one to consider equalities (2.6) as a system of algebraic relations connecting the basic new unknowns, the coefficients of the internal series (2.7). This proposal restricts the domain of applicability of representation (2.5) with very small times. Actually, the non-zero curvature of the shock wave itself is the function which changes rapidly with time and appears in Eq. (2.6), proving to have the greatest effect on the change in $\delta\eta_{i+1}/\delta t$. The use of series (2.7) leads to the fact that the curvature is a constant quantity which is determined by the geometry of the loaded surface. This is associated with the fact that, in the case of a non-zero curvature of the wave front, the power functions (2.7) do not always reflect the dynamics of the change in η_i even in the case of small post-impact times.

We next consider another version of the ray method. If it is assumed that the problem has a weak non-linearity, then it would be expected that its solution must only differ insignificantly from the solution of the analogous linearized problem. In system (2.6), we therefore replace the unknown function η_{k+2} with the solution of the linearized problem for the corresponding step. The function η_{k+2} is easily found in the linear case. Then, in other respects, when account in taken of the assumption which has been made, system (2.6) will be a closed system for determining of the quantities $\eta_1, \eta_2, \ldots, \eta_{i+1}$.

We will now consider the implementation of this scheme using the example of a longitudinal shock wave diverging from the surface $r = r_0$. For the corresponding linear problem, we obtain

$$\eta_1^L = \frac{\eta_{10}}{s}, \quad \eta_2^L = \frac{\eta_{20}}{s} - \frac{3C\eta_{10}(s^2-1)}{8r_0 s^3}, \quad s = \sqrt{1+\frac{Ct}{r_0}}$$
$$\eta_1^L(0) = \eta_{10} = -\upsilon_0, \quad \eta_2^L(0) = \eta_{20} = -a \tag{2.8}$$

For a quadratic law of motion $L(t)$, it suffices to take the first two terms in the ray expansion (2.5). This enables us to reduce the ray method to the representation of the initial equation of motion (2.1) in the discontinuities on the surface $\Sigma(t)$

$$\frac{\delta\eta_1}{\delta t} = \frac{-2\beta_1\eta_2\eta_1C^{-1} - G(t)\eta_1 r_\Sigma^{-1}(t) - \alpha_4\eta_1^2 r_\Sigma^{-1}(t)}{2(1-\gamma\eta_1 C^{-1})} + \ldots$$
$$\gamma = 1 + 7\beta_1/2 \tag{2.9}$$

The terms with a higher order of non-linearity which have not been written out are denoted by dots. Quantities occur in Eq. (2.9) which vary over a wide range from $G(t) \gg 1$ to $\eta_1 C^{-1} \ll 1$ as well as the function $r_\Sigma(t)$, which is also unknown. We assume that $\eta_2 \approx \eta_2^L$ and supplement Eqs. (2.9) with the equation

$$\frac{\delta r_\Sigma}{\delta t} = G(t) \tag{2.10}$$

in which G depends on η_1 in accordance with relations (1.4) and (2.3).

System (2.9), (2.10) can either be solved numerically or using the a small parameter method. We shall well on the last method. We define a small parameter $\varepsilon = \eta_{10}C^{-1}$, use the dimensionless variable s introduced above and consider the dimensionless unknown function $\omega(s) = \eta_1\eta_{10}^{-1}$ which we represent by an asymptotic series in powers of the small parameter:

$$\omega(s) = \omega_0(s) + \varepsilon\omega_1(s) + \varepsilon^2\omega_2(s) + \ldots$$

Substituting this series into Eqs. (2.9) and (2.10) we can obtain

$$\omega \approx \omega_0 + \varepsilon\omega_1 = \frac{1}{s} + \varepsilon\left\{\frac{-\beta_1 A(s^2-1)}{s} - \frac{\varphi}{s^2} - \frac{\beta_1}{s^3} - \frac{\gamma+\alpha_4}{s}\right\}$$
$$r_\Sigma \approx r_0 + Ct + 2\varepsilon r_0\beta_1(1-s)$$
$$\varphi = -\gamma - \beta_1 - \alpha_4, \quad A = \eta_{20}r_0/(\eta_{10}C) \tag{2.11}$$

It is assumed here that $A \sim 1$, which agrees with the possible scales of the characteristic quantities of the problem. The function $\omega_0(s)$ corresponds to the solution of the linear problem and ω_1 is the correction to it. Solution (2.11) holds up to the domain where $s \sim \varepsilon^{-1/4}$. If $s \sim \varepsilon^{-1/4}$, then series (2.11) loses uniformity and an additional expansion is required. Here, we are limited to the scale $s \sim 1$.

An approximate function $t_\Sigma(r)$ can be obtained by inversion of the series for $r_\Sigma(t)$

$$t_\Sigma = \frac{1}{C}\left\{r - r_0 + 2r_0\frac{\eta_{10}}{C}\beta_1\left(\sqrt{\frac{r}{r_0}} - 1\right)\right\} + \ldots \tag{2.12}$$

We now return to the representation of the function $u_r(r;\, t)$ by series (2.5). Substituting expressions (2.11) and (2.12) into it, we obtain

$$u_r = -\eta_{10}\left\{\frac{1}{\sqrt{H(r)}} + \frac{\eta_{10}}{C}\left[\frac{\beta_1\eta_{20}I(r)}{\eta_{10}\sqrt{H(r)}} - \frac{\varphi}{H(r)} - \frac{\beta_1}{H^{3/2}(r)} - \frac{\gamma+\alpha_4}{\sqrt{H(r)}}\right]\right\}(t+I(r)) -$$

$$-\frac{\eta_{20}}{2}\left\{\frac{1}{\sqrt{H(r)}} + \frac{3C^2\eta_{10}I(r)}{8r_0^2\eta_{20}H^{3/2}(r)}\right\}(t+I(r))^2 - \ldots$$

$$H(r) = \frac{r}{r_0} + 2\varepsilon\beta_1\left(\sqrt{\frac{r}{r_0}} - 1\right), \quad I(r) = \frac{r_0}{C}(1-H(r)) \tag{2.13}$$

To determine the constant η_{10}, η_{20}, it is necessary to substitute solution (2.13) into boundary condition (2.2). The structure of solution (2.13) and the approximate technique used to obtain it do not allow the boundary condition to be exactly satisfied. If the smallness of the post-impact time is taken into consideration, then, on substituting the expression for $r_L(t)$ into solution (2.3) and expanding the result in a Taylor series for small times, we find

$$\eta_{10} = -\frac{\upsilon_0}{1-\upsilon_0 C^{-1}}$$

$$\eta_{20} = \left\{-a - \frac{\upsilon_0^2}{r_0} - \frac{\upsilon_0 C^{-1}}{2}\left(a - \frac{\upsilon_0^2}{r_0}(-1+\gamma+\alpha_4)\right)\right\}\frac{1}{1+2\upsilon_0 C^{-1}} + \ldots \tag{2.14}$$

The next-front solution obtained in this way can be used independently as well as being included in numerical computational schemes in which the problem of the separation of the surface of a shock wave is posed. Actually, in this case, relations (2.12) and (2.13) can be looked at as formulae in which, for the first steps in time in the numerical procedure, the parameters η_{10} and η_{20} are determined from equalities (2.14). Thereafter, η_{10} and η_{20} can be considered as parameters with values which are improved during the course of the solution. A numerical computational procedure, based on the method of finite differences, is used to determine the displacement field and the deformations in the domain far from the shock wave. This method has been considered earlier.[13,14] We also note that there is no need to confine ourselves to the first two terms in solution (2.13). It is easy to construct the solution of the corresponding linear problem by the ray method up to an arbitrary k-th order. The replacement of η_k by the linear analogue enables us to reduce the error which is introduced into the solution by the linear approximation for η_2. The system of non-linear differential equations for $\eta_1, \eta_2, \ldots, \eta_{k-1}$ can also be solved using the small parameter method. The method can also be extended to multidimensional problems of impact deformation with the difference that, in these problems, it is necessary to determine the geometry of the ray coordinates simultaneously. The latter fact introduces certain complications but it does change the substance of the proposed method.

Acknowledgements

This research was financed by the Russian Foundation for Basic Research and the Far Eastern Division of the Russian Academy of Sciences (06-01-96005, 06-III-A-01-009).

References

1. Rossikhin YuA, Shitikova MV. Ray method for solving dynamic problems connected with propagation of wave surfaces of strong and weak discontinuities. *Appl Mech Rev* 1995;**48**(1):1–39.
2. Rossikhin YuA. Impact of a rigid sphere on an elastic half-space. *Prikl Mekh* 1986;**22**(5):15–21.
3. Podil'chuk Yu N, Rubtsov Yu K. Use of the method of ray series to investigate axi-symmetric unsteady problems in the dynamic theory of elasticity. *Prikl Mekh* 1986;**22**(3):3–9.
4. Burenin AA, Rossikhin Yu A. Effect of viscosity on the nature of the propagation of a plane longitudinal shock wave. *Zh Prikl Mekh Tekhn Fiz* 1990;**6**:13–7.
5. Bykovtsev GI, Vlasova IA. Special lines and surfaces in three-dimensional flows of ideal rigid-plastic media. In: *The Dynamics of a Deformable Solid (Continuous Dynamics). Issue 41.* Novosibirsk: Izd Inst Gidrodinamiki Sib Otd Akad Nauk SSSR; 1979, 31–6.
6. Burenin AA, Rossikhin YuA. A ray method for the solving of one-dimensional problems in the non-linear dynamic theory of elasticity with plane surfaces of strong discontinuities. In: *Applied Problems of the Mechanics of Deformable media.* Vladivostock: DVO Akad Nauk SSSR; 1991, 129–37.
7. Burenin AA. A possibility for the constructing of approximate solutions of unsteady problems in the dynamics of elastic media with impact effects. *Dal'nevostoch Mat Sbornik* 1999;**8**:49–72.
8. Bland DR. *Nonlinear Dynamic Elasticity. L. etc.*. Waltham, MA: Blaisdell; 1969.
9. McConnel AJ. *Applications of Tensor Analysis*. New York: Dover; 1957.
10. Kulikovskii AG, Sveshnikova EI. *Non-linear Waves in Elastic Media*. Moscow: Mosk Litsei; 1998.
11. Gerasimenko YeA, Ragozina VYe. Ray expansions in the study of the regularities of the propagation of non-planar shock waves. *Vestn SamGU Yestestvenoonauch Seriya* 2006;**6/1**(46):94–113.
12. Grinfel'd MA. *Methods of Continuous Mechanics in the Theory of Phase Transformations*. Moscow: Nauka; 1990.
13. Burenin AA, Zinov'ev PV. The Problem of the separation of the discontinuity surfaces in numerical methods of the dynamics of deformable media. In: *Problems of Mechanics. Collection of Papers on the Occasion of the 90th Birthday of A. Yu. Ishlinskii.* Moscow:Fizmatlit; 2003:146-55.
14. Burenin AA, Zinov'ev PV, Ragozina V Ye. Separation of discontinuity surfaces by the ray method in problems of the dynamics of elastic media. In: *Collection of Papers presented at the Intern. Scientific Conf. on Fundamental and Applied Problems of Mechanics.* Khabarovsk:Izd KhGTU;2003:**1**:62-4.

Translated by E.L.S.